

# Internal Space-time Symmetries of Massless Particles and Neutrino Polarization as a Consequence of Gauge Invariance

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## Abstract

There are gauge-transformation operators applicable to massless spin-1/2 particles within the little-group framework of internal space-time symmetries of massive and massless particles. It is shown that two of the  $SL(2, c)$  spinors are invariant under gauge transformations while the remaining two are not. The Dirac equation contains only the gauge-invariant spinors leading to polarized neutrinos. It is shown that the gauge-dependent  $SL(2, c)$  spinor is the origin of the gauge dependence of electromagnetic four-potentials. It is noted also that, for spin-1/2 particles, the symmetry group for massless particles is an infinite-momentum/zero-mass limit of the symmetry group for massive particles.

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# 1 Introduction

The internal space-time symmetries of massive and massless particles are dictated by the little groups of the Poincaré group which are isomorphic to the three-dimensional rotation group and the two-dimensional Euclidean group respectively [1]. The little group is the maximal subgroup of the Lorentz group whose transformations leave the four-momentum of the particle invariant. Using the properties of these groups we would like to address the following questions.

On massless particles, there are still questions for which answers are not readily available. Why do spins have to be parallel or anti-parallel to the momentum? While photons have a gauge degree of freedom with two possible spin directions, why do massless neutrinos have only one spin direction without gauge degrees of freedom? The purpose of this note is to address these questions within the framework of Wigner's little groups of the Poincaré group [1].

The group of Lorentz transformations is generated by three rotation generators  $J_i$  and three boost generators  $K_i$ . They satisfy the commutation relations:

$$[J_i, J_j] = i\epsilon_{ijk}J_k, \quad [J_i, K_j] = i\epsilon_{ijk}K_k, \quad [K_i, K_j] = -i\epsilon_{ijk}J_k. \quad (1)$$

In studying space-time symmetries dictated by Wigner's little group, it is important to choose a particular value of the four-momentum. For a massive point particle, there is a Lorentz frame in which the particle is at rest. In this frame, the little group is the three-dimensional rotation group. This is the fundamental symmetry associated with the concept of spin.

For a massless particle, there is no Lorentz frame in which its momentum is zero. Thus we have to settle with a non-zero value of the momentum along one convenient direction. The three-parameter little group in this case is isomorphic to the  $E(2)$  group. The rotational degree of freedom corresponds to the helicity of the massless particle, while the translational degrees of freedom are gauge degrees of the massless particle [2, 3].

In this report, we discuss first the  $O(3)$ -like little group for a massive particle. We then study the  $E(2)$ -like little group for massless particles. The  $O(3)$ -like little group is applicable to a particle at rest. If the system is boosted along the  $z$  axis, the little group becomes a “Lorentz-boosted” rotation group, whose generators still satisfy the commutation relations for the rotation group. However, in the infinite-momentum/zero-mass limit, the commutation relation should become those for massless particles. This process can be carried out for both spin-1 and spin-1/2 particles. This “group-contraction” process in principle can be extended all higher-spin particles. In this report, we are particularly interested in spin-1/2 particles.

In Sec. 2, we study the contraction of the  $O(3)$ -like little group to the  $E(2)$ -like little group. In Sec. 3, we discuss in detail the  $E(2)$ -like symmetry of massless particles. Secs. 4 and 5 are devoted to the question of neutrino polarizations and gauge transformation.

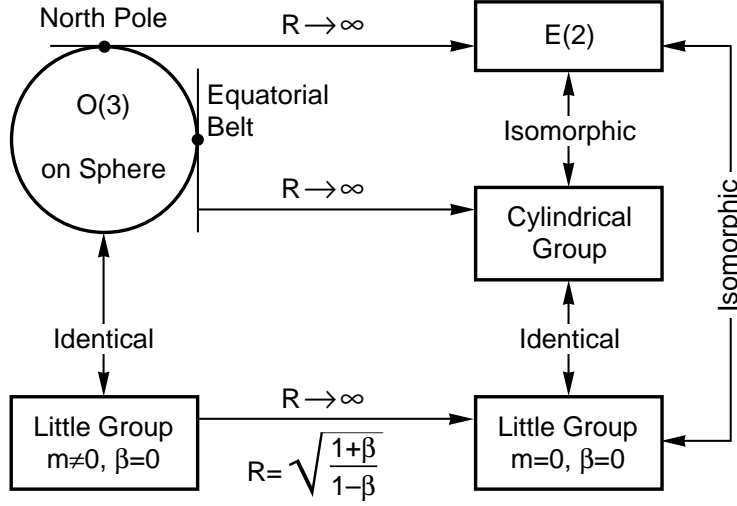


Figure 1: Contraction of  $O(3)$  to  $E(2)$  and to the cylindrical group, and contraction of the  $O(3)$ -like little group to the  $E(2)$ -like little group. The correspondence between  $E(2)$  and the  $E(2)$ -like little group is isomorphic but not identical. The cylindrical group is identical to the  $E(2)$ -like little group. The Lorentz boost of the  $O(3)$ -like little group for a massive particle is the same as the contraction of  $O(3)$  to the cylindrical group.

## 2 Massless Particle as a Limiting Case of Massive Particle

The  $O(3)$ -like little group for a particle at rest is generated by  $J_1, J_2$ , and  $J_3$ . If the particle is boosted along the  $z$  direction with the boost operator  $B(\eta) = \exp(-i\eta K_3)$ , the little group is generated by  $J'_i = B(\eta)J_iB(-\eta)$ . Because  $J_3$  commutes with  $K_3$ ,  $J_3$  remains invariant under this boost.  $J'_1$  and  $J'_2$  take the form

$$J'_1 = (\cosh \eta)J_1 + (\sinh \eta)K_2, \quad J'_2 = (\cosh \eta)J_2 - (\sinh \eta)K_1. \quad (2)$$

The boost parameter becomes infinite if the mass of the particle becomes vanishingly small. For large values of  $\eta$ , we can consider  $N_1$  and  $N_2$  defined as  $N_1 = -(\cosh \eta)^{-1}J'_2$  and  $N_2 = (\cosh \eta)^{-1}J'_1$  respectively. Then, in the infinite- $\eta$  limit,

$$N_1 = K_1 - J_2, \quad N_2 = K_2 + J_1. \quad (3)$$

These operators satisfy the commutation relations

$$[J_3, N_1] = iN_2, \quad [J_3, N_2] = -iN_1, \quad [N_1, N_2] = 0. \quad (4)$$

$J_3, N_1$ , and  $N_2$  are the generators of the  $E(2)$ -like little group for massless particles.

In order to relate the above little group to transformations more familiar to us, let us consider the two-dimensional  $xy$  coordinate system. We can make rotations around the

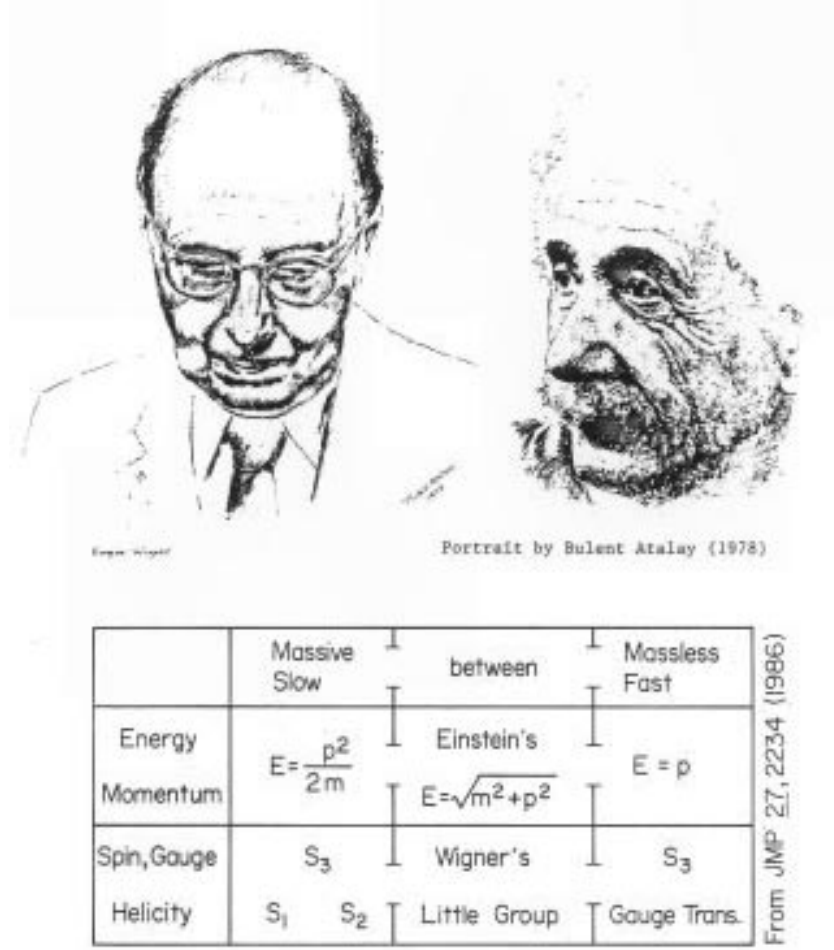


Figure 2: Symmetries of massive and massless particles. Wigner's little group unifies the internal space-time symmetries for massive and massless particles. We are interested in this paper how the symmetries of spin-1/2 particles can be unified.

origin and translations along the  $x$  and  $y$  axes. The rotation generator  $L_z$  takes the form

$$L_z = -i \left\{ x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right\} \quad (5)$$

The translation generators are

$$P_x = -i \frac{\partial}{\partial x}, \quad P_y = -i \frac{\partial}{\partial y}. \quad (6)$$

These generators satisfy the commutation relations:

$$[L_z, P_x] = iP_y, \quad [L_z, P_y] = -iP_x \quad [P_x, P_y] = 0. \quad (7)$$

These commutation relations are like those given in Eq.(4). They become identical if  $L_z$ ,  $P_x$  and  $P_y$  are replaced by  $J_1$ ,  $N_2$  and  $N_3$  respectively.

This group is not discussed often in physics, but is intimately related to our daily life. When we drive on the streets, we make translations and rotations, and thus make transformations of this  $E(2)$  group. Football players also make  $E(2)$  transformations in their fields.

As we shall see in the following section, the translation-like degrees of freedom in the  $E(2)$ -like little group lead to gauge transformations. Then Wigner's little group is like Einstein's  $E = mc^2$  which unifies the energy-momentum relations for both massive and massless particles. The little group unifies the internal space-time symmetries of massive and massless particles, as summarized in Fig. 2. The longitudinal rotation remains as the helicity degrees of freedom during the boost. However, the Lorentz boosted transverse rotations become gauge degrees of freedom, as summarized in Fig. 2.

The content of this table has been thoroughly studied in the literature [2]. Wigner's original paper was published in 1939 [1]. He worked out there the little groups for massive and massless particles. The task of obtaining the  $E(2)$ -like little group started in the 1953 paper by Inonu and Wigner [3], and the task was not completed until 1990 [2]. What was known in 1953 was the Euclidean group applicable to a flat surface, but the solution to the problem required also the cylindrical group hidden in the four-by-four Lorentz-transformation matrices. This has been thoroughly studied in the literature, and Fig 1 gives a complete picture of the contraction procedure.

### 3 Symmetry of Massless Particles

The internal space-time symmetry of massless particles is governed by the cylindrical group which is locally isomorphic to two-dimensional Euclidean group [2]. In this case, we can visualize a circular cylinder whose axis is parallel to the momentum. On the surface of this cylinder, we can rotate a point around the axis or translate along the direction of the axis. The rotational degree of freedom is associated with the helicity, while the translation corresponds to a gauge transformation in the case of photons.

The question then is whether this translational degree of freedom is shared by all massless particles, including neutrinos and gravitons. The purpose of this note is to show that the requirement of invariance under this symmetry leads to the polarization of neutrinos [4]. Since this translational degree of freedom is a gauge degree of freedom for photons, we can extend the concept of gauge transformations to all massless particles including neutrinos.

If we use the four-vector convention  $x^\mu = (x, y, z, t)$ , the generators of rotations around and boosts along the  $z$  axis take the form

$$J_3 = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad K_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \end{pmatrix}, \quad (8)$$

respectively. The remaining four generators are readily available in the literature [5]. They are applicable also to the four-potential of the electromagnetic field or to a massive vector meson.

The role of  $J_3$  is well known [6]. It is the helicity operator and generates rotations around the momentum. The  $N_1$  and  $N_2$  matrices take the form [5]

$$N_1 = \begin{pmatrix} 0 & 0 & -i & i \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, \quad N_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & i \\ 0 & i & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}. \quad (9)$$

The transformation matrix is

$$\begin{aligned} D(u, v) &= \exp \{-i(uN_1 + vN_2)\} \\ &= \begin{pmatrix} 1 & 0 & -u & u \\ 0 & 1 & -v & v \\ u & v & 1 - (u^2 + v^2)/2 & (u^2 + v^2)/2 \\ u & v & -(u^2 + v^2)/2 & 1 + (u^2 + v^2)/2 \end{pmatrix}. \end{aligned} \quad (10)$$

If this matrix is applied to the electromagnetic wave propagating along the  $z$  direction

$$A^\mu(z, t) = (A_1, A_2, A_3, A_0)e^{i\omega(z-t)}, \quad (11)$$

which satisfies the Lorentz condition  $A_3 = A_0$ , the  $D(u, v)$  matrix can be reduced to [4]

$$D(u, v) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ u & v & 1 & 0 \\ u & v & 0 & 1 \end{pmatrix}. \quad (12)$$

While  $A_3 = A_0$ , the four-vector  $(A_1, A_2, A_3, A_0)$  can be written as

$$(A_1, A_2, A_3, A_0) = (A_1, A_2, 0, 0) + \lambda(0, 0, \omega, \omega), \quad (13)$$

with  $A_3 = \lambda\omega$ . The four-vector  $(0, 0, \omega, \omega)$  represents the four-momentum. If the  $D$  matrix of Eq.(12) is applied to the above four vector, the result is

$$(A_1, A_2, A_3, A_0) = (A_1, A_2, 0, 0) + \lambda'(0, 0, \omega, \omega), \quad (14)$$

with  $\lambda' = \lambda + (1/\omega)(uA_1 + vA_3)$ . Thus the  $D$  matrix performs a gauge transformation when applied to the electromagnetic wave propagating along the  $z$  direction [4, 7, 8].

With the simplified form of the  $D$  matrix in Eq.(12), it is possible to give a geometrical interpretation of the little group [9]. If we take into account of the rotation around the  $z$  axis, the most general form of the little group transformations is  $R(\phi)D(u, v)$ , where  $R(\phi)$  is the rotation matrix. The transformation matrix is

$$R(\phi)D(u, v) = \begin{pmatrix} \cos \phi & -\sin \phi & 0 & 0 \\ \sin \phi & \cos \phi & 0 & 0 \\ u & v & 1 & 0 \\ u & v & 0 & 1 \end{pmatrix}. \quad (15)$$

Since the third and fourth rows are identical to each other, we can consider the three-dimensional space  $(x, y, z, z)$ . It is clear that  $R(\phi)$  performs a rotation around the  $z$  axis. The  $D$  matrix performs translations along the  $z$  axis. Indeed, the internal space-time symmetry of massless particles is that of the circular cylinder [2].

## 4 Massless Spin-1/2 Particles

The question then is whether we can carry out the same procedure for spin-1/2 massless particles. We can also ask the question of whether it is possible to combine two spin 1/2 particles to construct a gauge-dependent four-potential. With this point in mind, let us go back to the commutation relations of Eq.(1). They are invariant under the sign change of the boost operators. Therefore, if there is a representation of the Lorentz group generated by  $J_i$  and  $K_i$ , it is possible to construct a representation with  $J_i$  and  $-K_i$ . For spin-1/2 particles, rotations are generated by

$$J_i = \frac{1}{2}\sigma_i, \quad (16)$$

and the boosts by

$$K_i = (+)\frac{i}{2}\sigma_i, \quad \text{or} \quad K_i = (-)\frac{i}{2}\sigma_i. \quad (17)$$

The Lorentz group in this representation is often called  $SL(2, c)$ .

If we take the (+) sign, the  $N_1$  and  $N_2$  generators are

$$N_1 = \begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix}, \quad N_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \quad (18)$$

On the other hand, for the (-) sign, we use the “dotted representation” for  $N_1$  and  $N_2$ :

$$\dot{N}_1 = \begin{pmatrix} 0 & 0 \\ -i & 0 \end{pmatrix}, \quad \dot{N}_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \quad (19)$$

There are therefore two different  $D$  matrices:

$$D(u, v) = \begin{pmatrix} 1 & u - iv \\ 0 & 1 \end{pmatrix}, \quad \dot{D}(u, v) = \begin{pmatrix} 1 & 0 \\ -u - iv & 1 \end{pmatrix}. \quad (20)$$

These are the gauge transformation matrices for massless spin-1/2 particles [4, 5].

As for the spinors, let us start with a massive particle at rest, and the usual normalized Pauli spinors  $\chi_+$  and  $\chi_-$  for the spin in the positive and negative  $z$  directions respectively. If we take into account Lorentz boosts, there are two additional spinors. We shall use the notation  $\chi_{\pm}$  to which the boost generators  $K_i = (+)\frac{i}{2}\sigma_i$  are applicable, and  $\dot{\chi}_{\pm}$  to which  $K_i = (-)\frac{i}{2}\sigma_i$  are applicable. There are therefore four independent spinors [4, 5]. The  $SL(2, c)$  spinors are gauge-invariant in the sense that

$$D(u, v)\chi_+ = \chi_+, \quad \dot{D}(u, v)\dot{\chi}_- = \dot{\chi}_-. \quad (21)$$

On the other hand, the  $SL(2, c)$  spinors are gauge-dependent in the sense that

$$\begin{aligned} D(u, v)\chi_- &= \chi_- + (u - iv)\chi_+, \\ \dot{D}(u, v)\dot{\chi}_+ &= \dot{\chi}_+ - (u + iv)\dot{\chi}_-. \end{aligned} \quad (22)$$

The gauge-invariant spinors of Eq.(21) appear as neutrinos in the real world. The Dirac equation for massless neutrinos contains only the gauge-invariant  $SL(2, c)$  spinors.

## 5 The Origin of Gauge Degrees of Freedom

However, where do the above gauge-dependent spinors stand in the physics of spin-1/2 particles? Are they really responsible for the gauge dependence of electromagnetic four-potentials when we construct a four-vector by taking a bilinear combination of spinors?

The relation between the  $SL(2, c)$  spinors and the four-vectors has been discussed in the literature for massive particles [5, 10, 11]. However, is it true for the massless case? The central issue is again the gauge transformation. The four-potentials are gauge dependent, while the spinors allowed in the Dirac equation are gauge-invariant. Therefore, it is not possible to construct four-potentials from the Dirac spinors. However, it is possible to construct the four-vector with the four  $SL(2, c)$  spinors [4, 5]. Indeed,

$$\begin{aligned} -\chi_+\dot{\chi}_+ &= (1, i, 0, 0), & \chi_-\dot{\chi}_- &= (1, -i, 0, 0), \\ \chi_+\dot{\chi}_- &= (0, 0, 1, 1), & \chi_-\dot{\chi}_+ &= (0, 0, 1, -1). \end{aligned} \quad (23)$$

These unit vectors in one Lorentz frame are not the unit vectors in other frames. The  $D$  transformation applicable to the above four-vectors is clearly  $D(u, v)\dot{D}(u, v)$ .

$$\begin{aligned} D(u, v)\dot{D}(u, v)|\chi_+\dot{\chi}_+ &=> \dot{\chi}|\chi_+\dot{\chi}_+ > -(u + iv)|\chi_+\dot{\chi}_- >, \\ D(u, v)\dot{D}(u, v)|\chi_-\dot{\chi}_- &=> \chi_-\dot{\chi}_- > -(u + iv)|\chi_+\dot{\chi}_- >, \\ D(u, v)\dot{D}(u, v)|\chi_+\dot{\chi}_- &=> |\chi_+\dot{\chi}_- >. \end{aligned} \quad (24)$$

The component  $\chi_-\dot{\chi}_+ = (0, 0, 1, -1)$  vanishes from the Lorentz condition. The first two equations of the above expression correspond to the gauge transformations on the photon polarization vectors. The third equation describes the effect of the  $D$  transformation on the four-momentum, confirming the fact that  $D(u, v)$  is an element of the little group. The above operation is identical to that of the four-by-four  $D$  matrix of Eq.(12) on photon polarization vectors.

It is possible to construct a six-component Maxwell tensor by making combinations of two undotted and dotted spinors [5]. For massless particles, the only gauge-invariant components are  $|\chi_+\chi_+ >$  and  $|\dot{\chi}_-\dot{\chi}_- >$  [5]. They correspond to the photons in the Maxwell tensor representation with positive and negative helicities respectively.

It is also possible to construct Maxwell-tensor fields with for a massive particle, and obtain massless Maxwell fields by group contraction [12]. The construction of Maxwell tensors has been thoroughly studied by Weinberg in 1964 [13].

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